Convergence in Infinite Series of Positive Reciprocal Powers with Natural Numbers

Claudio H. González-Rojas, PhD.
Laboratorio de Computación Cuántica, Melipilla,
P.O. Box 9580000, RM, Chile, chgonzalezr@academicos.uta.cl

Abstract

The following is an example of numerical exploration applied to studying convergence in an infinite series of reciprocal powers. The numerical exploration consists of examining how much the result of the series increases relative to the initial value for a set of infinite series whose initial values begin with natural numbers such as 1/2, 1/3, 1/4, 1/5, and 1/6, respectively, and their generalization.

The examination of each series consisted of evaluating the convergence test, the series' limit value, and the series' growth ratio compared to the initial value. The convergence test was accompanied by the development of an algorithm written in R language, with iteration up to the term n=20.

The results obtained show that the power series studied are all convergent, positive, and have final values that are rational and irrational numbers less than one. Furthermore, each infinite series increases concerning the initial value by a maximum of 1/2 and a minimum of 1/5 regarding the first value of the respective series.

Keywords: infinite series, reciprocal power, convergent, rate of increase
Type of Article

Micro review: A well-established scientific result shown in an original perspective

Introduction

It is common knowledge that many infinite series with natural numbers converge to transcendental values such as π or e and are of the type

$$\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k} = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+.....+= ln(2) \qquad /1/$$

$$\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{(2k-1)} = 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+...+= \frac{\pi}{4} \qquad /2/$$

$$\sum_{k=0}^{\infty}\frac{1}{k!} = 1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+...+= e \qquad /3/$$

These are all widely disseminated in specialized literature (1-6) and correspond to infinite series with natural numbers (7) without the use of reciprocal powers. For example, the infinite series represented for equation /2/ is Leibniz's formula for π (8) .

The infinite series of reciprocal powers (8-14) also exist, which show convergence to fractional values of π^2, one of which is:

$$\sum_{k=0}^{\infty}\frac{1}{k^2} = 1+\frac{1}{2^2}+\frac{1}{3^2}+ \frac{1}{4^2}+\frac{1}{5^2}+....= \frac{\pi^2}{6} \qquad /4/$$

What could the convergence value be for the following infinite series in reciprocal power series, starting at 1/2?

$$\sum_{k=1}^{\infty}\frac{1}{2^k} = \frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+ \frac{1}{2^5}+...+\frac{1}{2^n} \; = ? \qquad /5/$$

Series with these characteristics have already been studied in the literature. Examples include the series:

$$\sum_{k=1}^{\infty}\frac{1}{i^n} \qquad /6/$$

Known as the Riemann zeta function (15-17) , it can be used to assess the constant π.

This constant, along with e and several other derivatives of infinite reciprocal series, could have interesting mathematical significance, particularly in astrophysics and string theory, as will be discussed later.

Revisiting the question posed in this paper, does this series of reciprocal powers, monotonically increasing, converge as the infinite sum grows from the second reciprocal and successive order onwards after the value of 1/2? The answer can be approached by starting with:

$$S_n = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \ldots + \frac{1}{2^n} \qquad /7/$$

Briefly examining the convergence test for this series. This requires that:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = r \qquad /8/$$

In this case:

$$\lim_{n\to\infty} \left|\frac{\frac{1}{2^{n+1}}}{\frac{1}{2^n}}\right| = \lim_{n\to\infty} \left|\frac{2^n}{2^{n+1}}\right| = \frac{1}{2} \qquad /9/$$

and ½ < 1, therefore the infinite series converges
Then, the following is composed

$$\frac{1}{2}S_n = \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \ldots + \frac{1}{2^{n+1}} \qquad /10/$$

Next, subtract:

$$S_n - \frac{1}{2}S_n = \frac{1}{2} - \frac{1}{2^{n+1}} \qquad /11/$$

$$\frac{1}{2}S_n = \frac{1}{2} - \frac{1}{2^{n+1}} \qquad /12/$$

Leading to:

$$S_n = 1 - \frac{1}{2^n} \qquad /13/$$

And taking the limit, when n -> ∞

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(1 - \frac{1}{2^n}\right) = 1 \qquad /14/$$

That is, the convergence limit of the series reaches an upper bound equal to 1.

Following, the increase in the limit value relative to the initial value is examined using the following ratio

$$\frac{limit\ value}{initial\ value} = \frac{S_2}{a_0} = \frac{1}{\frac{1}{2}} = \frac{2}{1} \qquad /15/$$

This allows the following to be inferred: in this monotonically increasing series, with an initial value of 1/2, it reaches a limit value, which is twice the initial value.

If the initial value of the series were 1/3, how much more would the infinite series increase with the sum of the reciprocal powers, starting from this initial value? The question extends to these other infinite series with initial values of 1/4, 1/5, 1/6, and others. For example, in the following series:

$$\sum_{k=1}^{\infty} \frac{1}{3^k} = \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \frac{1}{3^4} + \frac{1}{3^5} + \ldots + \frac{1}{3^n}$$

$$\sum_{k=1}^{\infty} \frac{1}{4^k} = \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \frac{1}{4^4} + \frac{1}{4^5} + \ldots + \frac{1}{4^n}$$

$$\sum_{k=1}^{\infty} \frac{1}{5^k} = \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \frac{1}{5^4} + \frac{1}{5^5} + \ldots + \frac{1}{5^n}$$

$$\sum_{k=1}^{\infty} \frac{1}{6^k} = \frac{1}{6} + \frac{1}{6^2} + \frac{1}{6^3} + \frac{1}{6^4} + \frac{1}{6^5} + \ldots + \frac{1}{6^n}$$

The following is an example of numerical exploration in natural numbers concerning a series of positive fractional powers.

Problem-solving

$$S_n = \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \frac{1}{3^4} + \frac{1}{3^5} + \ldots + \frac{1}{3^n}$$

Convergency test:

$$\lim_{n\to\infty} \left| \frac{\frac{1}{3^{n+1}}}{\frac{1}{3^n}} \right| = \lim_{n\to\infty} \left| \frac{3^n}{3^{n+1}} \right| = \frac{1}{3}$$

And 1/3 < 1 hence the infinite series converges

Subsequently, the following is composed.

$$\frac{1}{3}S_n = \frac{1}{3^2} + \frac{1}{3^3} + \frac{1}{3^4} + \frac{1}{3^5} + \ldots + \frac{1}{3^{n+1}}$$

Afterwards, subtract

$$S_n - \frac{1}{3}S_n = \frac{1}{3} - \frac{1}{3^{n+1}}$$

$$\frac{2}{3}S_n = \frac{1}{3} - \frac{1}{3^{n+1}}$$

Leading to:

$$S_n = \frac{1}{2} - \frac{1}{2 \cdot 3^n}$$

And taking the limit, when n -> ∞

$$\lim_{n \to \infty} S_n = \lim_{n \to \infty} \left(\frac{1}{2} - \frac{1}{2 \cdot 3^n}\right) = \frac{1}{2}$$

It can, therefore, be inferred that the upper bound of the infinite series, which begins with the value 1/3 and continues with the reciprocal powers of the same series, reaches a maximum value of 1/2. How much does the limit value increase compared to the initial value?

$$\frac{limit\ value}{initial\ value} = \frac{\frac{1}{2}}{\frac{1}{3}} = \frac{3}{2}$$

From this, it can be inferred that the initial value of series 1/3 grows, at most, one and a half times the initial value.

Following the same procedure for the series presented and the previous ones, it is possible to arrive at the results compiled in the following table.

Table I. Convergence test, initial series value, convergence limit, and growth ratio for the series under study

	Convergence test	a_0	Convergence limit	Growth ratio
$\sum_{k=1}^{\infty} \frac{1}{2^k}$	$\lim_{n\to\infty} \left\lvert \frac{2^n}{2^{n+1}} \right\rvert = \frac{1}{2}$	$\frac{1}{2}$	$S_2 = \lim_{n\to\infty} \left(1 - \frac{1}{2^n}\right) = 1$	$\frac{1}{\frac{1}{2}} = 2$
$\sum_{k=1}^{\infty} \frac{1}{3^k}$	$\lim_{n\to\infty} \left\lvert \frac{3^n}{3^{n+1}} \right\rvert = \frac{1}{3}$	$\frac{1}{3}$	$S_3 = \lim_{n\to\infty} \left(\frac{1}{2} - \frac{1}{2 \cdot 3^n}\right) = \frac{1}{2}$	$\frac{\frac{1}{2}}{\frac{1}{3}} = \frac{3}{2}$
$\sum_{k=1}^{\infty} \frac{1}{4^k}$	$\lim_{n\to\infty} \left\lvert \frac{4^n}{4^{n+1}} \right\rvert = \frac{1}{4}$	$\frac{1}{4}$	$S_4 = \lim_{n\to\infty} \left(\frac{1}{3} - \frac{1}{3 \cdot 4^n}\right) = \frac{1}{3}$	$\frac{\frac{1}{3}}{\frac{1}{4}} = \frac{4}{3}$
$\sum_{k=1}^{\infty} \frac{1}{5^k}$	$\lim_{n\to\infty} \left\lvert \frac{5^n}{5^{n+1}} \right\rvert = \frac{1}{5}$	$\frac{1}{5}$	$S_5 = \lim_{n\to\infty} \left(\frac{1}{4} - \frac{1}{4 \cdot 5^n}\right) = \frac{1}{4}$	$\frac{\frac{1}{4}}{\frac{1}{5}} = \frac{5}{4}$
$\sum_{k=1}^{\infty} \frac{1}{6^k}$	$\lim_{n\to\infty} \left\lvert \frac{6^n}{6^{n+1}} \right\rvert = \frac{1}{6}$	$\frac{1}{6}$	$S_6 = \lim_{n\to\infty} \left(\frac{1}{5} - \frac{1}{5 \cdot 6^n}\right) = \frac{1}{5}$	$\frac{\frac{1}{5}}{\frac{1}{6}} = \frac{6}{5}$

These empirical results could be generalized by:

$$S_n = \frac{1}{m} + \frac{1}{m^2} + \frac{1}{m^3} + \frac{1}{m^4} + \frac{1}{m^5} + \ldots + \frac{1}{m^n}$$

Then, the following is composed:

$$\frac{1}{m} S_n = \frac{1}{m^2} + \frac{1}{m^3} + \frac{1}{m^4} + \frac{1}{m^5} + \ldots + \frac{1}{m^{n+1}}$$

This is followed by subtracting

$$S_n - \frac{1}{m} S_n = \frac{1}{m} - \frac{1}{m^{n+1}}$$

$$S_n \left(1 - \frac{1}{m}\right) = \frac{1}{m} - \frac{1}{m^{n+1}}$$

This leads to:

$$S_n = \frac{\left(1 - \frac{1}{m^n}\right)}{(m-1)}$$

And taking the limit, when n -> ∞

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(\frac{\left(1 - \frac{1}{m^n}\right)}{(m-1)} \right) = \frac{1}{(m-1)} \;, m \neq 1$$

The following proposal, which has already been put forward in the literature (18) and is very similar to the present derivation, works as follows:

$$S_n = \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \frac{1}{x^4} + \frac{1}{x^5} + \ldots + \frac{1}{x^n}$$

And composing with

$$xS_n = 1 + \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \frac{1}{x^4} + \ldots + \frac{1}{x^{n-1}}$$

Leads to:

$$S_n = \frac{\left(1 - \frac{1}{x^n}\right)}{(x-1)}$$

This is precisely the same as the present proposal, with a similar algebraic approximation. The convergence calculated for all the series under study is shown graphically below; see Figure 1. The iteration is up to cycle 20. The algorithm was developed in R, using Rstudio (19), and is presented below.

```
# Calculus of infinite series
n <- 20
k <- 2  # k = 2,3,4,5,6
s1 <- 0
x <- c()
y <- c()
for (i in 1:n){
  s <- 1/((k))^(i)
  s1 <- s1 + s
  x1 <- i
  x <-c(x,x1)
  y <-c(y,s1)
}
plot(x,y, type="o", col="blue",
main="Calculus of serie", xlab="n", ylab="Value of S", xlim=c(1,n),
ylim=c(0.5,1.0))
```

Further interest in developing computational algorithms to assess convergence and other series parameters can also be found in (20-23). Table I summarizes the results of this iteration.

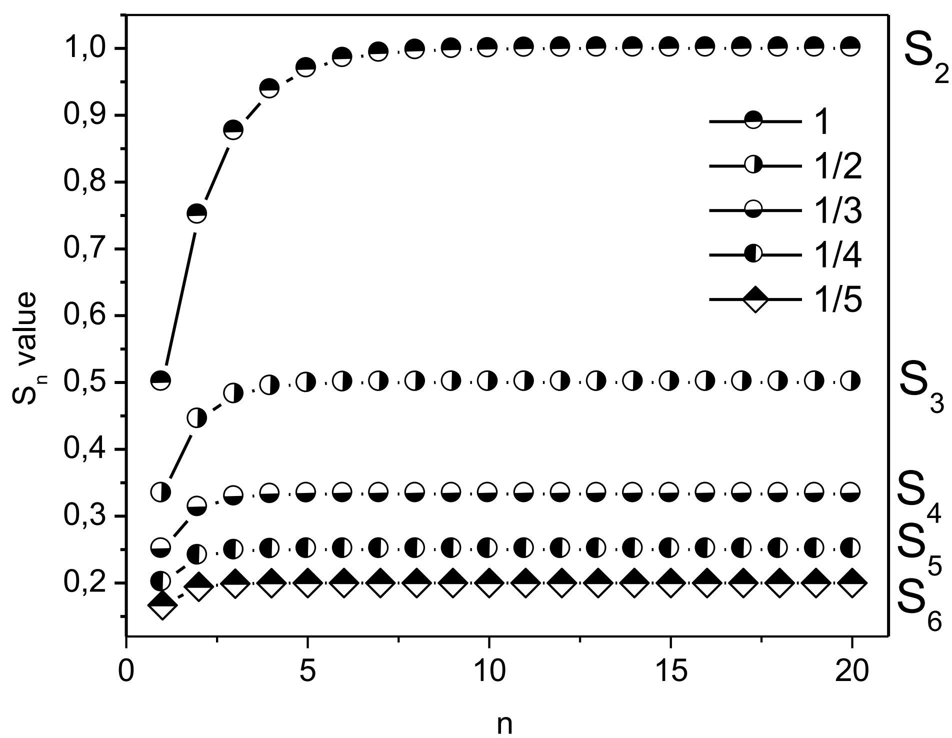

Figure 1. Convergence of infinite series with reciprocal exponents starting from initial values of 1/2, 1/3, 1/4, 1/5, and 1/6

Table I. Numerical values of the infinite series under study for the first 20 terms

n	S_2	S_3	S_4	S_5	S_6
1	0.5	0.333333333	0.25	0.2	0.166666667
2	0.75	0.444444444	0.3125	0.24	0.194444444
3	0.875	0.481481481	0.328125	0.248	0.199074074
4	0.9375	0.49382716	0.33203125	0.2496	0.199845679
5	0.96875	0.497942387	0.333007813	0.24992	0.19997428
6	0.984375	0.499314129	0.333251953	0.249984	0.199995713
7	0.9921875	0.499771376	0.333312988	0.2499968	0.199999286
8	0.99609375	0.499923792	0.333328247	0.24999936	0.199999881
9	0.998046875	0.499974597	0.333332062	0.249999872	0.19999998
10	0.999023438	0.499991532	0.333333015	0.249999974	0.199999997
11	0.999511719	0.499997177	0.333333254	0.249999995	0.199999999
12	0.999755859	0.499999059	0.333333313	0.249999999	0.2
13	0.999877930	0.499999686	0.333333328	0.25	0.2
14	0.999938965	0.499999895	0.333333332	0.25	0.2
15	0.999969482	0.499999965	0.333333333	0.25	0.2
16	0.999984741	0.499999988	0.333333333	0.25	0.2
17	0.999992371	0.499999996	0.333333333	0.25	0.2
18	0.999996185	0.499999999	0.333333333	0.25	0.2
19	0.999998093	0.5	0.333333333	0.25	0.2
20	0.999999046	0.5	0.333333333	0.25	0.2

Figure 1 and Table I show that convergence to the limit value is achieved between the fifth and tenth iterations. The convergence value for these infinite series of reciprocal powers lies in the interval (0.2,1.0). All the series under study are positive, converging to a single value, and the convergence value is a rational or irrational number.

Furthermore, at the beginning of this paper, it was mentioned that many infinite series are related to transcendental numbers in mathematics, such as Euler's number e or π. (24) Perhaps transcendence is consolidated with other results in other specialties such as astrophysics and string theory (25), as a reciprocal interaction between mathematics and natural laws (26). Upon reviewing the literature, it is possible to find Kempner's infinite series, whose infinite sum is S = 22. 9206766192641, divided by 34, which is a Fibonacci number, gives a value of 0.674137; those of twin primes, whose infinite sum is S = 1.902160583104, this number divided by 3, which is also a Fibonacci number, gives a value of 0.634; the number of palindromic prime numbers, whose infinite sum is S = 1.32398214680585, which when divided by 2, which is a Fibonacci number, gives a value of 0.6615 (27). The range is as follows: 0.63 – 0.67. In other words, this entire sequence of unrelated results are values that are close, in the context of astrophysics, to the spin of the final black hole produced by the collision of two black holes and calculated from observations of gravitational waves (28-30).

Meanwhile, the range of the series studied is 0.2 – 1.0. Therefore, all of them could be possible constants related to other observables verified in natural laws.

Conclusions

The infinite reciprocal power series studied are all convergent and positive and exhibit final rational and irrational values, less than unity.
The increase of each infinite series concerning the initial value increases by a maximum of 1/2 and a minimum of 1/6 compared to the first value of the respective series.
The convergence values represent possible constants related to other observables verified in natural laws.

Acknowledgement

The author, CGR, acknowledges **Laboratorio de Computación Cuántica** for dedicating time and lending the facilities during the development of the present work.

References

1. Bottazzi G. Sequences and Series. In: Bottazzi G, editor. Advanced Calculus for Economics and Finance: Theory and Methods. Cham: Springer International Publishing; 2023. p. 81-124.
2. Hardy G. Some theorems concerning infinite series. Mathematische Annalen. 1907;64(1):77-94.
3. Hardy GH. Divergent series: American Mathematical Society; 2024.
4. Khan RA. Convergence-Divergence of p-Series. The College Mathematics Journal. 2001;32(3):206-8.
5. Ponnusamy S. Series: Convergence and Divergence. In: Ponnusamy S, editor. Foundations of Mathematical Analysis. Boston, MA: Birkhäuser Boston; 2012. p. 147-207.
6. Rudin W. Nonanalytic functions of absolutely convergent Fourier series. Proceedings of the National Academy of Sciences. 1955;41(4):238-40.
7. Apostol TM. Another Elementary Proof of Euler's Formula for ζ(2n). The American Mathematical Monthly. 1973;80(4):425-31.
8. Series of reciprocals of a quadratic polynomial.
9. What is the infinite sum of reciprocals of all positive odd integers?
10. On the sums of series of reciprocals.
11. (https://math.stackexchange.com/users/255884/jg-mr-chapb) jmc. Sum of reciprocals of the perfect powers.
12. Borsos B, Kovács A, Tihanyi N. Tight upper and lower bounds for the reciprocal sum of Proth primes. The Ramanujan Journal. 2022;59(1):181-98.
13. Ing. Pier Franz Roggero DMN, 2, P.i. Francesco Di Noto. Sum of the reciprocals of famous series: mathematical connections with some sectors of theoretical physics and string theory. 2016.
14. Křížek M, Luca F, Somer L. On the convergence of series of reciprocals of primes related to the Fermat numbers. Journal of Number Theory. 2002;97(1):95-112.
15. Ivic A. The Riemann zeta-function: theory and applications: Courier Corporation; 2012.
16. Jessen B, Wintner A. Distribution functions and the Riemann zeta function. Transactions of the American Mathematical Society. 1935;38(1):48-88.
17. Karatsuba AA, Voronin SM. The Riemann zeta-function: Walter de Gruyter; 2011.
18. Find-value-of-sum-of-reciprocals-of-powers-of-a-number. Dec 7, 2014.
19. Horton NJ, Kleinman K. Using R and RStudio for data management, statistical analysis, and graphics. 2015.
20. Borwein JM, Bailey DH, Girgensohn R. Experimentation in mathematics: Computational paths to discovery: AK Peters/CRC Press; 2004.
21. Karatsuba EAe, editor Fast computation of transcendental functions. Doklady Akademii Nauk; 1991: Russian Academy of Sciences.
22. Kung HT. On computing reciprocals of power series. Numerische Mathematik. 1974;22(5):341-8.
23. Schönhage A. Variations on computing reciprocals of power series. Information Processing Letters. 2000;74(1-2):41-6.

24. Kline M. Euler and infinite series. Mathematics Magazine. 1983;56(5):307-15.
25. Schwarz JH, Seiberg N. String theory, supersymmetry, unification, and all that. Reviews of Modern Physics. 1999;71(2):S112.
26. Mukhi S. The reciprocal interaction between mathematics and natural law. Current Science. 2005;88(3):366.
27. Ing. Pier Franz Roggero DMN, P.i. Francesco Di Noto. . Sum of the reciprocals of famous series: mathematical connections with some sectors of theoretical physics and string theory. 2016., op cit. 13.
28. Pretorius F, East WE. Black hole formation from the collision of plane-fronted gravitational waves. Physical Review D. 2018;98(8):084053.
29. Reynolds CS. Observing black holes spin. Nature Astronomy. 2019;3(1):41-7.
30. Collaboration LS, Martynov D, Hall E, Abbott B, Abbott R, Abbott T. Publisher's Note: Sensitivity of the Advanced LIGO detectors at the beginning of gravitational wave astronomy [Phys. Rev. D 93, 112004 (2016)]. Phys Rev D Phys Rev D. 2016;97:059901.